Indian Statistical Institute, Bangalore
M. Math. Second Year Second Semester
Operator Theory

Back Paper Examination Date:
Maximum marks: 100 Time: 3 hours

(1) Let x, y be vectors in a Hilbert space $\mathcal{H}$. Show that
$$\langle x, y\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + e^{i\theta}y\|^2 e^{i\theta} d\theta.$$
[10]

(2) Suppose Z is a normal operator on a Hilbert space, show that $\|Z\|$ is equal to the spectral radius of Z.
[15]

(3) Let $\mathcal{A}$ be a unital Banach algebra. For $x \in \mathcal{A}$, let $\sigma(x)$ denote the spectrum of x. Show that for any complex polynomial p, $\sigma(p(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}$. [15]

(4) Let $C[0,1]$ be the C^*-algebra of continuous functions on the interval $[0,1]$ with supremum norm. Define $\phi : C[0,1] \to \mathbb{C}$ by
$$\phi(f) = \frac{1}{3}(f(0) + f(\frac{1}{2}) + f(1)).$$
Show that ϕ is a state. Obtain its GNS triple. What is the dimension of the associated Hilbert space? [15]

(5) Let A, B be $n\times n$ normal matrices. Suppose there exists an invertible matrix M such that $B = MAM^{-1}$. Show that there exists a unitary matrix U such that $B = UAU^{-1}$. (Hint: You may use polar decomposition.) [15]

(6) Let P be a non-trivial projection on a Hilbert space $\mathcal{H}$. Obtain the commutant and double commutant of the singleton $\{P\}$. [15]

(7) Let A be a positive operator on a separable Hilbert space $\mathcal{H}$, satisfying, $0 \leq A \leq I$. Let E be the spectral measure associated with A and let P be the projection $E([0, \frac{1}{2}])$. Obtain the spectral measure of AP. Show that AP is a positive operator satisfying $0 \leq AP \leq \frac{1}{2}$. [20]